Reprinted from
Proceedings of the Physical Society, Vol. 86, Part 1, No. 549, pp. 242–244
1965
Printed in Great Britain by J. W. Arrowsmith Ltd., Bristol 3

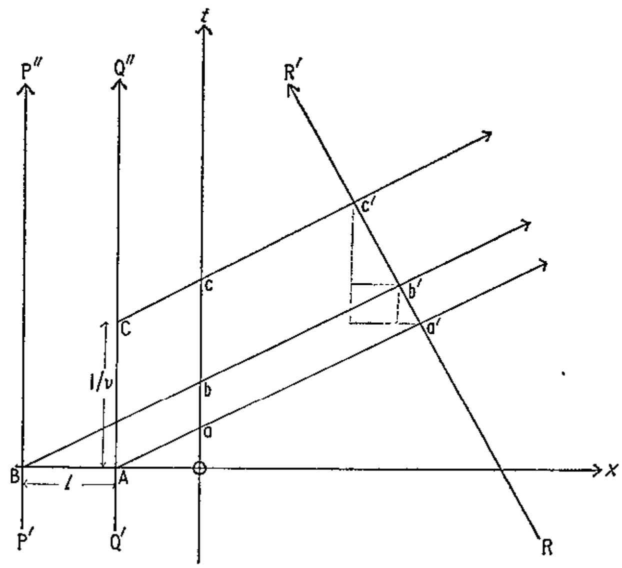
Further comments on Paper by R. Fürth, 'Proposal for an experiment to test the simultaneity theorem of the special theory of relativity'

I. Note on the invariance of the phase difference between two waves

Abstract. It is shown from first principles that the phase difference between two waves is independent of the motion of the observer. The different errors in two earlier treatments of the problem by Fürth are pointed out.

Fürth (1965 a, b) and Wormald (1965) have recently put forward conflicting views concerning the invariance of the phase difference between two waves. Although Wormald's argument is correct, he has not given an explicit demonstration of this invariance; in this letter we prove from first principles that it holds not only under a Lorentz transformation but also classically, and not only for electromagnetic waves but also for waves with any velocity. Further, Wormald does not point out the error in Fürth's original paper; here we do this, and also note the (different) error in Fürth's reply to Wormald's comment.

The physical situation is that of two transmitters at relative rest, separated by a distance l in their rest frame, emitting waves of the same frequency ν, and synchronized in the sense that they emit wave crests simultaneously in their rest frame. Only observers moving along the line joining the two transmitters are considered.



The figure is a space–time diagram for this situation, with events labelled by the coordinates x and t used by an observer stationary with respect to the transmitters P and Q. P′P″ and Q′Q″ are the world lines of the transmitters, the t axis is the world line of an observer stationary relative to them, and RR′ is the world line of an observer moving with velocity v relative to them. The events A and B, simultaneous with respect to P and Q, represent the emission of crests from P and Q, and C is the emission of the next crest from Q. These three crests are received by the stationary observer at events a, b and c respectively, and by the moving observer at a′, b′ and c′.

The phase difference δ between two waves as measured by any observer is 2π times the ratio of his proper time interval between the reception of crests from each of the two waves and his proper time interval between the reception of two successive crests from the same wave. The proper time between two events (x_1, t_1) and (x_2, t_2) for an observer moving uniformly from one to the other is an invariant function of the coordinate differences between the events, given in special relativity by

$$\Delta s_{12} = \left\{(t_2 - t_1)^2 - \frac{(x_2 - x_1)^2}{c^2}\right\}^{1/2}. \tag{1}$$

This expression, together with the definition of phase difference, means that for the stationary observer

$$\delta = \frac{2\pi\Delta s_{\mathrm{ab}}}{\Delta s_{\mathrm{ac}}} = \frac{2\pi(t_{\mathrm{b}} - t_{\mathrm{a}})}{t_{\mathrm{c}} - t_{\mathrm{a}}} = \frac{2\pi\nu l}{c}$$

while for the moving observer

$$\begin{aligned}\delta' &= \frac{2\pi\Delta s_{\mathrm{a'b'}}}{\Delta s_{\mathrm{a'c'}}} \\ &= 2\pi\left\{\frac{(t_{\mathrm{b'}} - t_{\mathrm{a'}})^2 - (x_{\mathrm{b'}} - x_{\mathrm{a'}})^2/c^2}{(t_{\mathrm{c'}} - t_{\mathrm{a'}})^2 - (x_{\mathrm{c'}} - x_{\mathrm{a'}})^2/c^2}\right\}^{1/2}.\end{aligned} \tag{2}$$

But, from the elementary geometry of the figure,

$$\frac{t_{\mathrm{b'}} - t_{\mathrm{a'}}}{t_{\mathrm{c'}} - t_{\mathrm{a'}}} = \frac{x_{\mathrm{b'}} - x_{\mathrm{a'}}}{x_{\mathrm{c'}} - x_{\mathrm{a'}}} = \frac{t_{\mathrm{b}} - t_{\mathrm{a}}}{t_{\mathrm{c}} - t_{\mathrm{a}}} = \frac{\delta}{2\pi}$$

so that (2) becomes simply

$$\delta' = \delta \tag{3}$$

and the phase difference is in fact invariant, as affirmed by Wormald.

If classical mechanics is used the relativistic relation (1) must be replaced by

$$\Delta s_{12}{}^{\mathrm{class}} = t_2 - t_1.$$

The subsequent reasoning is similar to that just given and the result (3) still holds.

We should also like to point out that the slope of the world lines Aa′, Bb′, and Cc′ of the three wave crests did not enter into the argument, which therefore holds for waves of any velocity (for example, sound waves).

In his original paper Fürth (1965 a) notes that according to the moving observer the transmitters are no longer synchronized in the sense defined above, and states correctly that this lack of synchrony will result in a shift

$$(\delta' - \delta)_1 = \frac{2\pi\nu l v}{c}$$

in the phase difference between the beams as seen by the two observers. He does not, however, state that there will be another shift, due to the fact that the difference in distance travelled by the two beams as seen by the two observers will not be the same; the size of this second shift is

$$(\delta' - \delta)_2 = -\frac{2\pi\nu l v}{c}$$

so that the total shift of phase difference is zero.

In his reply (Fürth 1965 b) to Wormald's comment Fürth gives a new treatment of the problem; this involves the shift $(\delta' - \delta)_2$, which is calculated exactly although the result is apparently regarded as only approximate. This time, however, the shift $(\delta' - \delta)_1$ (due to the asynchrony of the transmitters in the moving frame) is not considered, so that an erroneous result is again obtained, this time of opposite sign to that in the original paper.

School of Natural Philosophy,
University of St. Andrews.

M. V. Berry
R. C. Clark
P. J. Rijnierse

13th April 1965.

Fürth, R., 1965 a, *Proc. Phys. Soc.*, **85**, 131–3.
—— 1965 b, *Proc. Phys. Soc.*, **85**, 813.
Wormald, J. R., 1965, *Proc. Phys. Soc.*, **85**, 812.