

LETTER TO THE EDITOR

Remarks on degeneracies of semiclassical energy levels

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Abstract. Arnol'd showed that simple semiclassical methods applied to potentials with discrete symmetries yield 'quasimodes' which do not have the same degeneracy properties as true eigenstates; here it is shown that the true semiclassical states are simple linear combinations of the quasimodes.

Arnol'd (1972) proves under a plausible assumption that almost all quantum Hamiltonians invariant under rotation through 120° in the two-dimensional space of coordinates q_1, q_2 have some non-degenerate bound states, some degenerate bound states with multiplicity two and no degenerate bound states with higher multiplicity. He points out that on the other hand standard prescriptions for constructing semiclassical wavefunctions from tori in phase space surrounding stable closed classical orbits (Keller and Rubimow 1960, Maslov 1972, Lazutkin 1973) typically give energy levels with multiplicity three (an example is the semiclassical wavefunctions concentrated near the three 'heights' of a potential well whose boundary is an equilateral triangle with rounded corners). Therefore, Arnol'd argues, such semiclassical wavefunctions do not represent the classical limit of true quantum eigenstates but are only 'quasimodes', that is long-lived resonances.

Here I wish to show how semiclassical wavefunctions with the correct degeneracy properties can be constructed out of simple linear combinations of the quasimodes. Consider first the one-dimensional illustrative example of a symmetric double-well potential $V(q)$ (e.g. $V = q^4 - q^2$). For states below the energy of the central barrier the quasimodes have multiplicity two and can be represented as

$$(\psi_Q, 0) \quad \text{and} \quad (0, \psi_Q)$$

where (A, B) means that a wave of strength A inhabits the left-hand well and a wave of strength B inhabits the right-hand well. It is of course well known from WKB arguments that barrier penetration lifts the degeneracy, giving an energy splitting of order $\exp(-1/\hbar)$, with non-degenerate states

$$\left(\frac{\psi_Q}{\sqrt{2}}, \frac{\psi_Q}{\sqrt{2}}\right) \quad \text{and} \quad \left(\frac{\psi_Q}{\sqrt{2}}, -\frac{\psi_Q}{\sqrt{2}}\right),$$

the antisymmetric state having higher energy because of the extra node within the barrier.

In the two-dimensional potential with triangular symmetry the three quasimodes can be represented in an obvious notation as

$$\begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}.$$

Each quasimode is bounded by a caustic—the projection of the quantised torus in phase space. However, particles can leak from one quasimode into another through the ‘barriers’ between the caustics (by the quantum analogue of frustrated total internal reflection in optics). This will lead firstly to one symmetric state, namely

$$\begin{pmatrix} \psi_0/\sqrt{3} \\ \psi_0/\sqrt{3} \end{pmatrix},$$

and secondly to three unsymmetric states, namely

$$\begin{pmatrix} 2\psi_0/\sqrt{6} \\ -\psi_0/\sqrt{6} \end{pmatrix}, \quad \begin{pmatrix} -\psi_0/\sqrt{6} \\ 2\psi_0/\sqrt{6} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\psi_0/\sqrt{6} \\ -\psi_0/\sqrt{6} \end{pmatrix},$$

only two of which are independent because the sum of the three wavefunctions is zero; therefore the energy level of these three states is degenerate with multiplicity two, as expected on the basis of Arnol'd's general argument. These unsymmetric states should have higher energy than the non-degenerate symmetric state because of the extra nodes between the caustics.

It seems clear that in potentials with discrete symmetry all degenerate quasimodes formed by projection of tori which do not have the symmetry of the potential will communicate through barrier penetration between their caustics. Therefore they should live for times of order $\exp(+1/\hbar)$ before decaying into the true semiclassical eigenstates which have energy spacings of order $\exp(-1/\hbar)$.

Dr M Tabor has pointed out to me that quasimodes of a very different sort have been considered by Gutzwiller (1971). These are localised near unstable closed classical orbits surrounded by regions in phase space where motion is stochastic, and take the form of resonances whose energy width is of order \hbar . Therefore they should be relatively short-lived, decaying after a time of order \hbar^{-1} into the true semiclassical eigenstates which Voros (1976, 1977) and I (Berry 1977a,b) have conjectured will occupy the whole of the stochastic region (e.g. the whole energy surface if the motion is ergodic).

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References

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