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Riemann–Silberstein vortices for paraxial waves

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Abstract
Stationary lines of phase singularity for monochromatic electromagnetic fields can be defined as the zeros of a complex scalar field $\Psi(\mathbf{r})$ constructed from the real fields $\mathbf{E}_{\mathrm{real}} + \mathrm{i}\mathbf{B}_{\mathrm{real}}$. In terms of the Riemann–Silberstein vector $\mathbf{F} = \mathbf{E}_{\mathrm{real}} + \mathrm{i}\mathbf{B}_{\mathrm{real}}$, $\Psi(\mathbf{r})$ is the time average of $\mathbf{F}(\mathbf{r}, t) \cdot \mathbf{F}(\mathbf{r}, t)$. For paraxial waves, $\Psi(\mathbf{r})$ can be specified entirely in terms of the transverse complex electric (or magnetic) field and its spatial derivatives. The accuracy of the paraxial theory is illustrated numerically.

Keywords: polarization, singularities, vortices, paraxial

1. Introduction

In seminal papers, Nye and Hajnal [1, 2] reported the discovery of lines of singular polarization—purely circular and purely linear—for electric and magnetic fields in space. This research generalized their earlier studies [3–6] of the singularities of the transverse components of the fields with respect to a specified direction (e.g. of well-defined propagation). Because they refer to the electric and magnetic fields separately, these singularities are not invariant under Lorentz transformations. This does not compromise their applicability to situations with a natural rest frame, for example when sources, receivers, and diffracting objects are not in relative motion, or where waves propagate in a stationary refracting medium; in such cases, it is sensible to regard the electric and magnetic fields as separate.

However, it is interesting from a fundamental theoretical viewpoint to investigate relativistically invariant line singularities of the full electromagnetic field. Such singularities are the *Riemann–Silberstein vortices* studied recently [7] by Bialynicki-Birula and Bialynicka-Birula. In terms of the real electromagnetic fields $\mathbf{E}_{\mathrm{real}}(\mathbf{r}, t)$, $\mathbf{B}_{\mathrm{real}}(\mathbf{r}, t)$, where $\mathbf{r} = \{x, y, z\}$, these vortices are defined by

$$\mathbf{F}(\mathbf{r}, t) \cdot \mathbf{F}(\mathbf{r}, t) = 0, \tag{1}$$

where the complex scalar field $\mathbf{F}$ is the Riemann–Silberstein vector

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{E}_{\mathrm{real}}(\mathbf{r}, t) + \mathrm{i}c\mathbf{B}_{\mathrm{real}}(\mathbf{r}, t). \tag{2}$$

Zeros of $\mathbf{F} \cdot \mathbf{F}$ are lines in space, on which the phase $\arg \mathbf{F} \cdot \mathbf{F}$ is singular and around which the phase gradient vector circulates; in spacetime, the singularities are surfaces.

For the important case of general monochromatic waves (superpositions of plane waves with the same frequency ω), the advantage of Riemann–Silberstein vortices is lost, because monochromaticity is not preserved under Lorentz boosts: plane-wave components travelling in different directions get differently Doppler-shifted. Reflecting this, the Riemann–Silberstein vortices for general monochromatic waves oscillate with frequency 2ω, making them difficult to observe at optical frequencies. There are some exceptions, for example, Gauss–Laguerre beams, where the fields have spatial symmetry, and the recently-discovered important class of helicity states [8] (superpositions of purely left- or right-handed circularly polarized plane waves), for which the Riemann–Silberstein vortex lines are stationary. Nevertheless, for general monochromatic waves, the vortices move.

It is however possible to define stationary lines for general monochromatic waves: they are the zeros of the *time-average* $\langle \mathbf{F} \cdot \mathbf{F} \rangle_{\mathrm{time}}$. In terms of the complex fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, defined by

$$\mathbf{E}_{\mathrm{real}}(\mathbf{r}, t) = \mathrm{Re}(\mathbf{E}(\mathbf{r}) \exp(-\mathrm{i}\omega t)),$$
$$\mathbf{B}_{\mathrm{real}}(\mathbf{r}, t) = \mathrm{Re}(\mathbf{B}(\mathbf{r}) \exp(-\mathrm{i}\omega t)), \tag{3}$$

the time-averaged complex scalar field is

$$\Psi(\mathbf{r}) = 2\langle \mathbf{F} \cdot \mathbf{F} \rangle_{\mathrm{time}} = \mathbf{E}^* \cdot \mathbf{E} - c^2 \mathbf{B}^* \cdot \mathbf{B} + 2\mathrm{i}c \,\mathrm{Re}\, \mathbf{B}^* \cdot \mathbf{E}$$
$$= (\mathbf{E}^* + \mathrm{i}c\mathbf{B}^*) \cdot (\mathbf{E} + \mathrm{i}c\mathbf{B}). \tag{4}$$

My purpose here is to point out that the condition for the line zeros of $\Psi(\mathbf{r})$ takes a particularly interesting form for monochromatic *paraxial waves*. These are superpositions of plane waves whose directions span a small range of angles,

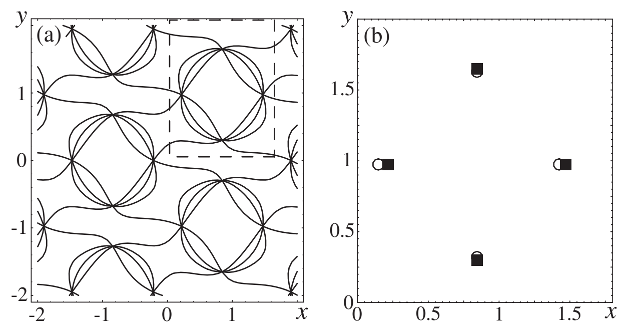

Figure 1. (a) Riemann–Silberstein vortices, shown as intersections of contours of arg $\Psi(x, y)$ (equation (4)) at intervals of $\pi/4$, for the three-wave electromagnetic superposition described in the text. (b) Magnification of dashed region of (a), showing vortices computed from the exact fields (filled squares) and from the approximation (11) (open circles).

centred on the z direction, say. If the spatial frequency is $k = 2\pi/\lambda = \omega/c$, paraxial waves take the form

$$\mathbf{E}(\mathbf{r}), \mathbf{B}(\mathbf{r}) = \exp(\mathrm{i}kz) \times \text{vectors varying slowly with } \mathbf{r}. \quad (5)$$

For paraxial waves, it is convenient to separate the transverse and longitudinal fields:

$$\mathbf{E} = \{\mathbf{E}_\mathrm{t}, E_z\}, \qquad \mathbf{B} = \{\mathbf{B}_\mathrm{t}, B_z\}, \qquad \text{where } \mathbf{E}_\mathrm{t} = \{E_x, E_y\}, \quad \mathbf{B}_\mathrm{t} = \{B_x, B_y\}. \quad (6)$$

Then Maxwell's equations enable the E_z, B_z, and the transverse magnetic field $\mathbf{B}_\mathrm{t}$, to be approximated, to lowest order in $1/k$, in terms of the transverse electric field $\mathbf{E}_\mathrm{t}$, leading to

$$E_z \approx \frac{\mathrm{i}}{k}\nabla_\mathrm{t}\cdot\mathbf{E}_\mathrm{t}, \qquad B_z \approx \frac{\mathrm{i}}{k}\nabla_\mathrm{t}\cdot\mathbf{B}_\mathrm{t},$$

$$B_z \approx \frac{\mathrm{i}}{kc}\left(-\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}\right) = \frac{-\mathrm{i}}{kc}\mathbf{e}_z\cdot\nabla_\mathrm{t}\times\mathbf{E}_\mathrm{t},$$

$$\mathbf{B}_\mathrm{t} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} \approx \frac{1}{c}\begin{pmatrix} -E_y - \frac{1}{2k^2}\left\{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)E_y - 2\frac{\partial^2}{\partial x\,\partial y}E_x\right\} \\ E_x - \frac{1}{2k^2}\left\{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)E_x + 2\frac{\partial^2}{\partial x\,\partial y}E_y\right\} \end{pmatrix}, \quad (7)$$

where $\nabla_\mathrm{t} = \{\partial/\partial x, \partial/\partial y\}$.

After some elementary manipulations of equation (4), the scalar field $\Psi(\mathbf{r})$ becomes

$$\Psi(\mathbf{r}) \approx \frac{1}{2k^2}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2\mathrm{i}\frac{\partial^2}{\partial x\,\partial y}\right) \times \left(|E_x|^2 - |E_y|^2 - 2\mathrm{i}\,\mathrm{Re}\,E_x^* E_y\right)\right]. \quad (8)$$

If the $1/k^2$ corrections are neglected, then $\Psi = 0$ and the entire field is 'vortical', rendering the concept nugatory; this is obvious from (4), since in this extreme paraxial limit $\mathbf{E}$ and $\mathbf{B}$ are transverse and orthogonal, and $|\mathbf{B}| = |\mathbf{E}|/c$.

With the corrections included, paraxial fields typically possess isolated Riemann–Silberstein vortex lines, that can be studied as points in the x, y plane. This is illustrated in figure 1(a), which is a phase map of $\Psi(\mathbf{r})$ for a superposition of three symmetrically arrayed plane waves whose directions make an angle 25° with the z axis, elliptically polarized with the same handedness and ellipticity 0.6, with one of the polarization ellipses perpendicular to the other two, computed exactly from the vanishing of (4). This is similar to a field nominally studied experimentally [2]; for an explicit expression, see [9]. The vortices are the intersections of the phase contours.

The paraxial field Ψ takes a particularly interesting form in terms of the circular components of the transverse electric field, namely

$$E_\mathrm{L}(\zeta, \bar\zeta) = \frac{1}{\sqrt{2}}(E_x + \mathrm{i}E_y), \qquad E_\mathrm{R}(\zeta, \bar\zeta) = \frac{1}{\sqrt{2}}(E_x - \mathrm{i}E_y), \quad (9)$$

written in terms of the complex coordinates

$$\zeta = x + \mathrm{i}y, \qquad \bar\zeta = x - \mathrm{i}y. \quad (10)$$

Then (8) can be written as

$$\Psi(\mathbf{r}) \approx \frac{4}{k^2}\frac{\partial^2}{\partial\bar\zeta^2}(E_\mathrm{L}^* E_\mathrm{R}). \quad (11)$$

Figure 1(b) shows how accurately several of the vortices from figure 1(a) are approximated by the vanishing of this paraxial expression, even though 25° is not a very small angle.

There is a mathematical analogy [10] between Riemann–Silberstein vortices and the circular polarization (C) lines of the electric or magnetic fields, because each sort of singularity is specified by the vanishing of a complex scalar field. But the vanishing of $\Psi(\mathbf{r})$ in (4) has no connection to circular polarization, and is satisfied identically for a plane wave. Similarly, there is a mathematical analogy [10] between the linear polarization (L) lines of the electric or magnetic fields, and the (non-Lorentz-invariant) vanishing of the Poynting vector $\mathbf{V} = \mathbf{E}_\mathrm{real}\times\mathbf{B}_\mathrm{real}$, because each sort of singularity defines lines by the condition that two real 3-vectors are parallel. But the lines $\mathbf{V} = 0$ have no connection with linear polarization, and the singularities defined by the time average $\langle\mathbf{V}\rangle = 0$ are surfaces for purely transverse waves (and do not exist at all in the approximation where $\mathbf{E}$ and $\mathbf{B}$ are orthogonal), and collapse to points when longitudinal fields are included.

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